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A HYBRID ITERATIVE APPROACH FOR SOLVING NONLINEAR TIME-FRACTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FRACTIONAL REACTION-TRANSPORT MODELS

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Abstract: This paper proposes a novel hybrid iterative method for the numerical solution of nonlinear fractional differential equations (FDEs) in the Liouville-Caputo sense. The methodology integrates the Formable integral transform with a new algorithm based on the Daftardar-Gejji and Jafari iterative method to provide accurate approximations for complex FDEs. The efficacy of the approach is demonstrated through applications to the chemical Schnakenberg model and the coupled one-dimensional time-fractional Keller-Segel chemotaxis model. Numerical results confirm the convergence of fractional-order solutions towards their corresponding integer-order formulations, thereby validating the precision and reliability of the proposed technique. This study contributes significantly to the computational analysis of fractional reaction-transport phenomena and offers novel insights into the dynamic characteristics of nonlinear fractional models.

Keywords and Phrases: Fractional reaction-diffusion model, Chemotaxis, Series solution, Formable integral transform, Iterative technique.

2020 Mathematics Subject Classification: 35R11, 35C10, 44A05, 35K57, 35Q92.

1. Introduction

Fractional calculus is a branch of mathematical analysis focused on derivatives and integrals of arbitrary order, extending the concepts of integer-order differentiation and multiple integrations. Unlike classical derivatives, the fractional differential operators exhibit nonlocal properties, which implies that a future state of the system depends on the entire past states of the system. This feature establishes it as a potent mathematical tool for modelling and analysing systems and processes that exhibit non-standard dynamical behaviour across various disciplines, including mathematics, engineering and physics. However, finding exact solutions to these nonlinear fractional equations using standard elementary functions can be highly complex, highlighting the need for an effective and reliable algorithm to solve such equations. Consequently, significant research has been focused on developing analytical and numerical methods to either approximate or characterize the solution of these intricate fractional models. Integral transform based homotopy methods, Adomian decomposition and variational iteration methods combined with integral transforms, residual power series method, finite difference schemes, spectral collocation method, hybrid numerical techniques are some of the notable emerging methods, that have been developed and actively used over the past few decades to solve fractional differential equations. For detailed view, one can refer [6, 8, 11].

In 2006, Daftardar-Gejji and Jafari presented an effective iterative approach called the New Iterative Method (NIM), also known as the Daftardar-Jafari method, for solving nonlinear and linear functional equations, including algebraic, integral, differential equations of integer and fractional orders, as well as for the system of equations. The Daftardar-Jafari method, is user-friendly and is easy to implement with any software, often yields better approximations than traditional numerical and analytical techniques for solving fractional differential equations. While Daftardar-Gejji and Jafari (DGJ) pioneered NIM, later researchers have combined the method with Laplace-like integral transforms to handle nonlinear problems, as these transformations can change the differential and integral equations into algebraic forms within the transformed domain. Some prominent methods include the iterative Laplace transform method [22] for Whitham-Broer-Kaup equations, semi analytical method for solving Boussinesq equation [24], the Elzaki iterative method [28] for the time-fractional Navier Stokes equation, iterative Aboodh transform method [15], A new general integral transform method [9], the extended natural transform with Daftardar-Jafari polynomials [21], Upadhyaya integral transform solving non-linear Volterra integral equations [10] and the Khalouta-Daftardar-

Jafari method [3]. However, these integral transforms have limitations when applied to complex nonlinear fractional models. For example, the Laplace transform involves inversion integrals that can reduce numerical stability, and the Elzaki and Natural transforms, although effective in certain linear or boundary-value problems, offer limited flexibility in handling nonlinearities.

The Formable integral transform (FT) overcomes several of these limitations, particularly for nonlinear fractional differential equations. Unlike the Laplace transform, it features a straightforward and stable inversion process, and it naturally integrates initial conditions as defined in the Liouville-Caputo fractional derivative framework, which is essential for accurately modeling systems with memory effects. Additionally, FT is highly compatible with iterative algorithms, enabling efficient treatment of nonlinear terms and improving convergence rates. Its algebraic properties simplify the manipulation of fractional operators, facilitating analytical and numerical solution techniques. These features make the Formable integral transform a robust and computationally efficient tool for nonlinear fractional reaction-transport models.

Motivated by these advantages, this work proposes the Formable transform iterative algorithm (FTIA), which synergistically combines the Formable integral transform with the updated version of the NIM as developed and detailed in [13], to construct series approximations of nonlinear fractional differential equations.

Originating from Rania Zohair Saadeh's innovative work [26], the Formable integral transform exhibits compatibility with other transforms, well-defined derivative properties, and supports convolution operations, making it highly applicable to a variety of mathematical problems. In this study, we apply FTIA to obtain approximate solutions for the following nonlinear fractional models:

1.1 Schnakenberg Model

In 1979, the German biophysicist Jurgen Schnakenberg introduced a system of two coupled partial differential equations, namely the Schnakenberg model, that describes an autocatalytic chemical reaction coupled with diffusion. The model typically involves two chemical species, often denoted as G_1 and G_2, and is based on a trimolecular reaction scheme with autocatalysis given by

$$A \rightleftharpoons G_1 \; B \to G_2, \; 2G_1 + G_2 \to 3G_1.$$

The time-fractional Schnakenberg (FS) model advances the classical model by substituting integer-order time derivatives with fractional derivatives. Due to its simplicity and its ability to exhibit Turing instability, FS model becomes a vibrant research domain, providing a more refined and potentially precise approach for modelling reaction-diffusion processes, particularly in systems displaying in systems displaying non-standard behaviors. Its applications range from investigating molecular movement within the cytoplasm, the dynamics of cells and signalling molecules during tumor invasion, the kinetics of drug release from polymeric reactions in complex polymer solutions, to examining pigmentation patterns in skin. Solving the chemical FS model typically involves a combination of analytical approximations and sophisticated numerical techniques tailored to handle the challenges posed by fractional derivatives and nonlinearities. The choice of method depends on the specific problem, the level of precision needed and the available computational resources. The Laplace Adomian decomposition method (LADM) [14], Quasilinearization technique [5], Natural transform decomposition technique (NDT) [4], Laplace residual power series method [12] are some of the prominent solution methods used to tackle the FS model. In this paper, we aim to utilize FTIA to approximate the solution of the following fractional chemical Schnakenberg model:

$$\begin{aligned} \frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= \mathsf{a} - g_1(x,t) + g_1^2(x,t)g_2(x,t) + \mathsf{D}_1\frac{\partial^{2\mu} g_1(x,t)}{\partial x^{2\mu}} \\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= \mathsf{b} - g_1^2(x,t)g_2(x,t) + \mathsf{D}_2\frac{\partial^{2\mu} g_2(x,t)}{\partial x^{2\mu}} \end{aligned} \tag{1.1}$$

subject to the initial conditions,

$$\begin{aligned} g_1(x,0) &= g_{10}(x), \\ g_2(x,0) &= g_{20}(x), \end{aligned} \tag{1.2}$$

where,

- $x \in \mathbb{R},\ t > 0$, denotes the space and time variables respectively
- $g_1(x,t)$ and $g_2(x,t)$ represents the concentration of the product G_1 and G_2 respectively
- the constants a and b reflects the concentration of the source A and B respectively
- D_1 and D_2 are coefficient of diffusion of the substances G_1 and G_2 respectively
- $0 < \lambda, \mu \leq 1$, denotes the order of fractional derivatives defined in Caputo's sense.

1.2 Keller-Segal Model

The Keller-Segel system of equations, which mathematically represents a set of nonlinear partial differential equations, describes the directed movement of cells influenced by a chemical process known as chemotaxis. The classical model was introduced by Lee Segel and Evelyn Keller through a parabolic system to investigate the aggregation of mould in cellular slime driven by chemical attraction. However, in various biological contexts, the chemoattractant process demonstrates anomalous diffusion instead of the typical Brownian motion, leading to the incorporation of fractional derivatives to model such movements, culminating in the fractional Keller-Segel (FKS) system, expressed as follows:

$$\begin{aligned} \frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= \mathsf{a}\ \frac{\partial^2 g_1(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(g_1(x,t).\frac{\partial \psi(g_2)}{\partial x}\right), \\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= \mathsf{b}\ \frac{\partial^2 g_2(x,t)}{\partial x^2} + \mathsf{c}\ g_1(x,t) - \mathsf{d}\ g_2(x,t), \end{aligned} \tag{1.3}$$

with initial conditions,

$$\begin{aligned} g_1(x,0) &= g_{10}(x), \\ g_2(x,0) &= g_{20}(x), \end{aligned} \tag{1.4}$$

where,

- $x \in \mathbb{R},\ t > 0$, denotes the space and time variables used to locate the position of the cells at given time t.
- $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ are all positive constants and $0 < \lambda \leq 1$, denote the order of fractional derivative defined in Caputo's sense
- $g_1(x,t)$ and $g_2(x,t)$ represents the intensity of cells and of the chemical signal that lures the cells respectively, at the position (x,t)
- the nonlinear term $\frac{\partial}{\partial x}\left(g_1(x,t).\frac{\partial \psi(g_2)}{\partial x}\right)$ represents the directed movement of cells induced by the chemical signal, and $\psi(g_2)$ denotes the sensitivity function.

As the FKS model seems to be effective in examining how cells cluster into specific patterns, it has been widely employed in the study of numerous biological processes including the movement of bacteria towards nutrient sources, the process of tumor angiogenesis and invasion, the migration of leukocytes during inflammatory responses and so on. Several robust and effective techniques such as homotopy perturbation Sumudu transform method [27], the Aboodh residual power series method [1], Yang decomposition method [29], Laplace homotopy analysis method [20], and so on have been put forth for solving FKS model.

In this paper, we explore three cases of FKS equations characterized by the chemotactic sensitivity function $\psi(g_2) = 1, \psi(g_2) = g_2$ and $\psi(g_2) = g_2^2$, respectively. The analysis is organized as follows: Section 2 introduces key definitions related to fractional derivatives and Formable integral transforms, along with relevant properties and theorems. Section 3 details the methodology employed for solving the specified nonlinear time-fractional differential equations using the FTIA. Section 4 demonstrates the effectiveness of our approach through the solution of four nonlinear fractional reaction-transport models. Section 5 provides a graphical discussion of the results, and finally, Section 6 concludes the analysis.

2. Essential Concepts

This section highlights the key principles of FC and FT, along with the basic definitions and necessary theorems in Caputo fractional sense, to be applied in subsequent sections.

Definition 2.1. [16] *The Riemann-Liouville time-fractional integral of order* $\lambda > 0$ *of a function* $g(x,t)$ *is defined as*

$$I_t^{\lambda}[g(x,t)] = \frac{\partial^{-\lambda} g}{\partial t^{-\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{(\lambda-1)} g(x,t)\, ds.$$

Definition 2.2. [16] *The Caputo time-fractional derivative of order* $0 < \lambda \leq 1$ *of a function* $g(x,t)$ *is defined as*

$$\frac{\partial^{\lambda} g}{\partial t^{\lambda}} = I_t^{1-\lambda}\left(\frac{\partial g}{\partial t}\right).$$

Definition 2.3. [16] *The Mittag-Leffler function of order* $\lambda > 0$ *of a real valued function* g *is defined as*

$$E_{\lambda}(g) = \sum_{k=0}^{\infty} \frac{g^k}{\Gamma(k\lambda+1)}, \ Re(\lambda) > 0.$$

Definition 2.4. *A real valued function* $g(t)$ *is said to be of exponential order* θ, *if* $\exists A > 0$, *such that* $|g(t)| \leq Ae^{\theta t}, \forall t > 0$.

Definition 2.5. *Let* $g(t)$ *be a piecewise continuous, exponential order function defined on the set*

$$\mathcal{M} = \{g(t)/\exists A \in (0,\infty), \theta_1, \theta_2 > 0, |g(t)| \leq Ae^{\frac{t}{\theta_i}}, \ if \ t \in [0,\infty)\},$$

then the Formable integral transform [26] of the function $g(t)$ is defined as

$$\mathsf{F}[g(t)] = \mathsf{B}(s,v) = s\int_0^\infty e^{-st}\ g(vt)\ dt,\ \ v \in (0,\infty).$$

with the respective inverse transform given by

$$\mathsf{F}^{-1}[\mathsf{B}(s,v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{\frac{st}{v}} \mathsf{B}(s,v)\ ds,\ \ v \in (0,\infty).$$

Theorem 2.1. *If g is a continuously differentiable function, $g \in C^q[a,b]$, $q \in \mathbb{N}$ and if λ denotes the order of fractional derivative, then for $q-1 < \lambda < q$,* [25]

$$\mathsf{F}\left[\frac{d^\lambda g(t)}{dt^\lambda}\right] = \frac{s^\lambda}{v^\lambda}\mathsf{F}[g(t)] - \sum_{k=0}^{q-1} \frac{s^{\lambda-k} g^{(k)}(0)}{v^{\lambda-k}}$$

In particular, for $0 < \lambda < 1$, we have,

$$\mathsf{F}\left[\frac{d^\lambda g(t)}{dt^\lambda}\right] = \frac{s^\lambda}{v^\lambda}\mathsf{F}[g(t)] - \frac{s^\lambda g(0)}{v^\lambda}.$$

3. FTIA: Methodology and Analysis

This section offers a clear overview of the proposed methodology, setting the stage for the subsequent analysis and for discussion of its implementation and results.

Consider the system of m FDEs

$$\frac{\partial^\lambda g_i(x,t)}{\partial t^\lambda} = \mathcal{H}_i\left(x,t\right) + \mathcal{L}_i\left[g_1, g_2, \cdots, g_m\right] + \mathcal{N}_i\left[g_1, g_2, \cdots, g_m\right], \quad 1 \leq i \leq m, \tag{3.1}$$

with initial conditions

$$g_i(x,0) = g_{i0}(x), \quad 0 < \lambda \leq 1,\ x \in \mathbb{R},\ t > 0,\ m \in \mathbb{N}, \tag{3.2}$$

where $\mathcal{N}$ and $\mathcal{L}$ denotes the non-linear and linear terms of the function $g_i(x,t)$ respectively.

To derive the functional equation form of eqn.(3.1), we take Formable integral transform on both sides of each equation of the system and then, by using definition 2.5 and theorem 2.1 along with initial conditions (3.2) we get,

$$\frac{s^\lambda}{v^\lambda}\left[\mathsf{F}[g_i(x,t)] - g_i(x,0)\right] = \mathsf{F}\left[\mathcal{H}_i\left(x,t\right) + \mathcal{L}_i\left[g_1, g_2, \cdots, g_m\right] + \mathcal{N}_i\left[g_1, g_2, \cdots, g_m\right]\right],$$

which implies,

$$\mathsf{F}[g_i(x,t)] = g_{i0}(x) + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathcal{H}_i\left(x,t\right)\right] + \mathsf{F}\left[\mathcal{L}_i\left[g_1, g_2, \cdots, g_m\right]\right] + \mathsf{F}\left[\mathcal{N}_i\left[g_1, g_2, \cdots, g_m\right]\right]\right]. \tag{3.3}$$

Taking inverse Formable integral transform on both sides of eqn.(3.3), and after simplification, we get a functional equation corresponding to the i^{th} equation of the system (3.1) as follows,

$$g_i(x,t) = \mathcal{H}_i^*(x,t) + \mathcal{L}_i^*[g_1, g_2, \cdots, g_m] + \mathcal{N}_i^*[g_1, g_2, \cdots, g_m], \tag{3.4}$$

where,

$$\mathcal{H}_i^*(x,t) = g_{i0}(x) + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\mathsf{F}\left[\mathcal{H}_i\left(x,t\right)\right]\right],$$
$$\mathcal{L}_i^*[g_1, g_2, \cdots, g_m] = \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\mathsf{F}\left[\mathcal{L}_i\left[g_1, g_2, \cdots, g_m\right]\right]\right],$$
$$\mathcal{N}_i^*[g_1, g_2, \cdots, g_m] = \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\mathsf{F}\left[\mathcal{N}_i\left[g_1, g_2, \cdots, g_m\right]\right]\right].$$

In accordance with the algorithm defined for series solution for the system of FDEs as referenced in [13], let us approximate the solution function (3.4) with an infinite series, $g_i(x,t) = \sum_{r=0}^{\infty} g_{ir}(x,t)$ for each $i, (\ 1 \leq i \leq m)$ and denote ν_{ik}, the k^{th} approximated series of the solution $g_i(x,t)$.

$$(i.e.) \quad \nu_{ik} = \sum_{r=0}^{k} g_{ir}(x,t),$$

whose terms are calculated through an iterative formula,

$$\nu_{i0} = g_{i0}(x,t) = \mathcal{H}_i^*(x,t), \tag{3.5}$$
$$\nu_{ik} = \sum_{r=0}^{k} g_{ir}(x,t) = \mathcal{H}_i^*(x,t) + \mathcal{L}_i^*[\nu_{ik-1}] + \mathcal{N}_i^*[\nu_{ik-1}], \text{ for } k \geq 1,\ 1 \leq i \leq m. \tag{3.6}$$

Under suitable conditions, the series $\nu_{ik} \to g_i(x,t), as\ k \to \infty$.

The convergence analysis of the proposed iterative scheme is rigorously established in the following theorem. Utilizing the framework of a general Banach space and the contractive property of the iterative operator, the sequence of approximations forms a Cauchy sequence, which converges to a unique fixed point in the complete normed space. This classical approach provides a rigorous and widely accepted theoretical foundation for the method's convergence without requiring specialized function spaces such as Sobolev spaces [7, 19] or advanced notions like statistical convergence [23]. Thus, the analysis balances mathematical rigor and clarity, ensuring robust convergence guarantees for the nonlinear fractional differential equations under consideration.

Theorem 3.1. *Condition for Convergence:*
Let $g_{ik}(x,t)$ be in a Banach space $\mathcal{B}$ for each $k \in \mathbb{Z}^+$ with $g_{i0}(x,t)$ bounded in $\mathcal{B}$. Then the sequence of iterated values ν_{ik} defined in (3.5)-(3.6) converges uniformly to the solution function $g_i(x,t)$, as $k \to \infty$, provided that there exists $0 < \eta < 1$ such that $\left\|g_{i(k+1)}(x,t)\right\| \leq \eta \left\|g_{ik}(x,t)\right\|$, for each i.

Proof. Assume that there exists $0 < \eta < 1$ such that $\left\|g_{i(k+1)}(x,t)\right\| \leq \eta \left\|g_{ik}(x,t)\right\|$. Then to prove the statement, it is enough to show that the sequence of partial sums $\{\nu_{ik}\}$ is a Cauchy sequence in $\mathcal{B}$ [2].
For $p, q \in \mathbb{N}$ with $p > q$, we have,

$$\begin{aligned}\left\|\nu_{ip} - \nu_{iq}\right\| &= \left\|\nu_{ip} - \nu_{i(p-1)} + \nu_{i(p-1)} - \nu_{i(p-2)} + \nu_{i(p-2)} - \cdots + \nu_{i(q+1)} - \nu_{iq}\right\| \\ &\leq \left\|\nu_{ip} - \nu_{i(p-1)}\right\| + \left\|\nu_{i(p-1)} - \nu_{i(p-2)}\right\| + \cdots + \left\|\nu_{i(q+1)} - \nu_{iq}\right\|\end{aligned}$$

But from the assumption, we have,
$\left\|\nu_{k+1} - \nu_k\right\| = \left\|g_{i(k+1)}(x,t)\right\| \leq \eta \left\|g_{ik}(x,t)\right\| \leq \eta^2 \left\|g_{i(k-1)}(\tilde{x},\tilde{t})\right\| \cdots \leq \eta^{k+1} \left\|g_{i0}(x,t)\right\|$.
Thus,

$$\begin{aligned}\left\|\nu_{ip} - \nu_{iq}\right\| &\leq \left[\eta^p + \eta^{(p-1)} + \cdots + \eta^{(q+1)}\right] \left\|g_{i0}(x,t)\right\|, \\ &= \eta^{q+1} \left[1 + \eta + \eta^2 + \cdots + \eta^{(p-q-1)}\right] \left\|g_{i0}(x,t)\right\|, \\ &= \eta^{q+1} \left(\frac{1 - \eta^{p-q}}{1 - \eta}\right) \left\|g_{i0}(x,t)\right\|, \\ &\leq \left(\frac{\eta^{q+1}}{1 - \eta}\right) \left\|g_{i0}(x,t)\right\|.\end{aligned}$$

Since, $g_{i0}(x,t)$ bounded in $\mathcal{B}$, $\left\|g_{i0}(x,t)\right\| < \infty$ and $\lim_{p,q\to\infty} \left\|\nu_{ip} - \nu_{iq}\right\| = 0$, which proves that the sequence $\{\nu_{ik}\}$ is a cauchy sequence in $\mathcal{B}$ and thus it converges in $\mathcal{B}$.

4. Illustrative Examples

This section uses FTIA to analyze four fractional initial value problems (IVPs), with all computations performed using Mathematica 14 software.

Example 4.1. Consider the time-fractional Schnakenberg model (1.1)-(1.2) for

$\mathsf{a} = 1, \mathsf{b} = 1, \mu = 1$, with diffusion coefficients $\mathsf{D}_1 = 2, \mathsf{D}_2 = 2$ along with the initial conditions $g_{10}(x) = e^x \sin(x)$ and $g_{20}(x) = xe^x$. This implies,

$$\begin{aligned} \frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= 1 - g_1(x,t) + g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_1(x,t)}{\partial x^2}, \\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= 1 - g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_2(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0,\ 0 < \lambda \leq 1 \end{aligned} \tag{4.1}$$

subject to the initial conditions,

$$\begin{aligned} g_1(x,0) &= e^x \sin(x), \\ g_2(x,0) &= xe^x. \end{aligned} \tag{4.2}$$

In the view of FTIA explained in section 3, taking Formable integral transform on both sides of (4.1), we get,

$$\begin{aligned} \mathsf{F}\left[\frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda}\right] &= \mathsf{F}\left[1 - g_1(x,t) + g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_1(x,t)}{\partial x^2}\right], \\ \mathsf{F}\left[\frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda}\right] &= \mathsf{F}\left[1 - g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_2(x,t)}{\partial x^2}\right]. \end{aligned}$$

Using definition 2.5 and theorem 2.1 along with initial conditions (4.2), we get,

$$\begin{aligned} \frac{s^\lambda}{v^\lambda}\left[\mathsf{F}\left[g_1(x,t)\right] - e^x \sin(x)\right] &= \mathsf{F}\left[1 - g_1(x,t) + g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_1(x,t)}{\partial x^2}\right], \\ \frac{s^\lambda}{v^\lambda}\left[\mathsf{F}\left[g_2(x,t)\right] - xe^x\right] &= \mathsf{F}\left[1 - g_1^2(x,t)g_2(x,t) + 2\frac{\partial^2 g_2(x,t)}{\partial x^2}\right], \end{aligned}$$

after simplification, we get,

$$\begin{aligned} \mathsf{F}\left[g_1(x,t)\right] &= e^x \sin(x) + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[1 - g_1(x,t) + 2\frac{\partial^2 g_1(x,t)}{\partial x^2}\right]\right] + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[g_1^2(x,t)g_2(x,t)\right]\right], \\ \mathsf{F}\left[g_2(x,t)\right] &= xe^x + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[1 + 2\frac{\partial^2 g_2(x,t)}{\partial x^2}\right]\right] + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[-g_1^2(x,t)g_2(x,t)\right]\right]. \end{aligned}$$

Now, by taking inverse Formable integral transform on both sides, we get,

$$\begin{aligned} g_1(x,t) &= e^x \sin(x) + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[1 - g_1(x,t) + 2\frac{\partial^2 g_1(x,t)}{\partial x^2}\right]\right]\right] \\ &\quad + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[g_1^2(x,t)g_2(x,t)\right]\right]\right] \\ &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[g_1, g_2] + \mathcal{N}_1^*[g_1, g_2], \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} g_2(x,t) &= xe^x + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[1+2\frac{\partial^2 g_2(x,t)}{\partial x^2}\right]\right]\right] + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[-g_1^2(x,t)g_2(x,t)\right]\right]\right] \\ &= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[g_1,g_2] + \mathcal{N}_2^*[g_1,g_2], \end{aligned} \tag{4.4}$$

Let us approximate $g_1(x,t)$ in (4.3) and $g_2(x,t)$ in (4.4) with an infinite series, $g_1(x,t) = \sum_{r=0}^{\infty} g_{1r}(x,t)$ and $g_2(x,t) = \sum_{r=0}^{\infty} g_{2r}(x,t)$.
Then by the proposed scheme, the series solution of the system (4.1)-(4.2) with $(k+1)$ approximated terms be given by $\nu_{ik} = \sum_{r=0}^{k} g_{ir}(x,t)$, $1 \le i \le 2$, whose terms are calculated using iterative scheme (3.5)-(3.6) as follows:

$$\begin{aligned} \nu_{10} &= \mathcal{H}_1^*(x,t) = e^x\sin(x), \\ \nu_{20} &= \mathcal{H}_2^*(x,t) = xe^x, \\ \nu_{11} &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[\nu_{10},\nu_{20}] + \mathcal{N}_1^*[\nu_{10},\nu_{20}] \\ &= e^x\sin(x) + \frac{t^\lambda\left(e^{3x}x\sin^2(x) - e^x\sin(x) + 4e^x\cos(x) + 1\right)}{\Gamma(\lambda+1)}, \\ \nu_{21} &= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[\nu_{10},\nu_{20}] + \mathcal{N}_2^*[\nu_{10},\nu_{20}] \\ &= e^x x + \frac{t^\lambda\left(2\left(e^x x + 2e^x\right) - e^{3x}x\sin^2(x) + 1\right)}{\Gamma(\lambda+1)}, \\ \nu_{12} &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[\nu_{11},\nu_{21}] + \mathcal{N}_1^*[\nu_{11},\nu_{21}] \\ &= e^x\sin(x) + \frac{t^\lambda\left(e^{3x}x\sin^2(x) - e^x\sin(x) + 4e^x\cos(x) + 1\right)}{\Gamma(\lambda+1)} \\ &\quad + \frac{t^{2\lambda}\begin{pmatrix} 2e^{5x}x^2\sin^3(x) - e^{5x}x\sin^4(x) + e^{2x}\left(e^x(13x+16)+1\right)\sin^2(x) \\ +e^x\left(2e^x x - 15\right)\sin(x) + 4e^{3x}x\cos^2(x) \\ +8e^x\left(e^{2x}(4x+1)\sin(x) - 1\right)\cos(x) - 1 \end{pmatrix}}{\Gamma(2\lambda+1)} + \cdots, \\ \nu_{22} &= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[\nu_{11},\nu_{21}] + \mathcal{N}_2^*[\nu_{11},\nu_{21}] \\ &= e^x x + \frac{t^\lambda\left(2\left(e^x x + 2e^x\right) - e^{3x}x\sin^2(x) + 1\right)}{\Gamma(\lambda+1)} \\ &\quad + \frac{e^x t^{2\lambda}\begin{pmatrix} 4(x+4) + e^{4x}x(\sin(x) - 2x)\sin^3(x) - e^x(2x+\sin(x))\sin(x) \\ +e^{2x}(-9x - 4(4x+1)\sin(2x) + (5x+8)\cos(2x) - 8) \end{pmatrix}}{\Gamma(2\lambda+1)} + \cdots, \\ &\ \vdots \end{aligned}$$

Hence, the approximate solution of the system (4.1)-(4.2) is given by

$$g_1(x,t) = e^x\sin(x) + \frac{t^\lambda\left(e^{3x}x\sin^2(x) - e^x\sin(x) + 4e^x\cos(x) + 1\right)}{\Gamma(\lambda+1)}$$

$$+ \frac{t^{2\lambda}\begin{pmatrix} 2e^{5x}x^2\sin^3(x) - e^{5x}x\sin^4(x) + e^{2x}\left(e^x(13x+16)+1\right)\sin^2(x) \\ +e^x\left(2e^xx - 15\right)\sin(x) + 4e^{3x}x\cos^2(x) \\ +8e^x\left(e^{2x}(4x+1)\sin(x) - 1\right)\cos(x) - 1 \end{pmatrix}}{\Gamma(2\lambda+1)}$$

$$+ \frac{t^{3\lambda}\Gamma(2\lambda+1)}{\Gamma(3\lambda+1)}\times$$

$$\left(\frac{\begin{pmatrix} 2e^{4x}\sin^2(x)\left(8x^2+\cos(2x)-1\right) + 4e^{7x}x(\sin(x)-2x)^2\sin^4(x) \\ +e^{5x}\sin(x)(32x(9x+7) - (193x+108)\sin(x) + 16x(23x+8)\sin(2x) \\ +9(3x+4)\sin(3x) - 8(14x+3)\cos(x) - 16(2x(5x+7)+1)\cos(2x) \\ +8(14x+3)\cos(3x)+16) - 4e^x(52\cos(x) - 47\sin(x)) \\ +2e^{2x}(16(x+2)\sin(x) + \cos(2x) + 16(2x+2\sin(x)+1)\cos(x)+7) \\ +2e^{3x}(263x + 8(61x+114)\sin(2x) + (713x+180)\cos(2x) + 492) + 4 \end{pmatrix}}{4\Gamma(2\lambda+1)} + \frac{\begin{pmatrix} e^x\left(x + \sin(x)\left(e^x(3x+8) + e^{3x}x\sin(x)(x-2\sin(x)) + 2\right) + 4e^xx\cos(x)\right) \\ \left(e^x\left(\sin(x)\left(e^{2x}x\sin(x)-1\right) + 4\cos(x)\right) + 1\right) \end{pmatrix}}{\Gamma(\lambda+1)^2} \right)$$

$$+\cdots, \tag{4.5}$$

$$g_2(x,t) = e^xx + \frac{t^\lambda\left(2\left(e^xx + 2e^x\right) - e^{3x}x\sin^2(x) + 1\right)}{\Gamma(\lambda+1)}$$

$$+ \frac{e^xt^{2\lambda}\begin{pmatrix} 4(x+4) + e^{4x}x(\sin(x)-2x)\sin^3(x) - e^x(2x+\sin(x))\sin(x) \\ +e^{2x}(-9x - 4(4x+1)\sin(2x) + (5x+8)\cos(2x) - 8) \end{pmatrix}}{\Gamma(2\lambda+1)}$$

$$+ \frac{e^xt^{3\lambda}}{\Gamma(\lambda+1)^2\Gamma(3\lambda+1)}\times$$

$$\begin{pmatrix} \Gamma(\lambda+1)^2\begin{pmatrix} e^{4x}\sin(x)\begin{pmatrix} -73x^2 - 56x - 4(23x+8)x\sin(2x) \\ +(49x+27)\sin(x) - (7x+9)\sin(3x) \\ +(28x+6)\cos(x) + (x(41x+56)+4)\cos(2x) \\ -2(14x+3)\cos(3x) - 4 \end{pmatrix} \\ +e^{3x}\left(\sin^2(x) - 4x^2\right)\sin^2(x) + 8x - e^{6x}x(\sin(x)-2x)^2\sin^4(x) \\ -2e^x((5x+8)\sin(x) + 4\sin(2x) + (8x+4)\cos(x) + 2) + 48 \\ -e^{2x}(149x + 16(17x+29)\sin(2x) + (347x+76)\cos(2x) + 260) \end{pmatrix} \\ -\left(x + \sin(x)\left(e^x(3x+8) + e^{3x}x\sin(x)(x-2\sin(x)) + 2\right) + 4e^xx\cos(x)\right) \\ \Gamma(2\lambda+1)\left(e^x\left(\sin(x)\left(e^{2x}x\sin(x)-1\right) + 4\cos(x)\right) + 1\right) \end{pmatrix}$$

$$+\cdots, \tag{4.6}$$

We observe that the resulting series matches well with the one given by LADM in [14] and NDT in [4]. Moreover, the solution of the classical chemical Schnakenberg model will be obtained by substituting $\lambda = 1$ in eqn. (4.5) and eqn.(4.6).

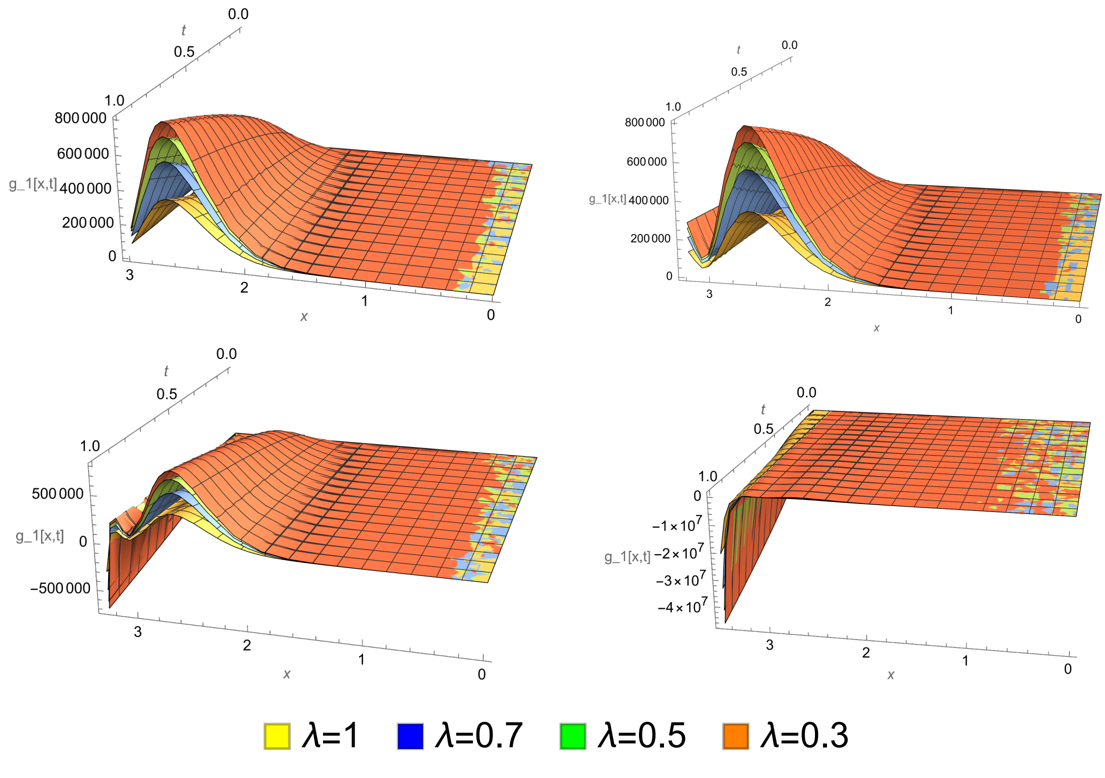

Figure 4.1: Approximate solution $g_1(x,t)$ of the system (4.1)-(4.2) in different spatial frames.

Example 4.2. Consider the time-FKS model (1.3)-(1.4) with sensitivity term $\psi(g_2) = 1$ along with the initial conditions $g_{10}(x) = m_1 e^{-x^2}$ and $g_{20}(x) = m_2 e^{-x^2}$. Then we have, $\frac{\partial}{\partial x}\left(g_1(x,t).\frac{\partial(1)}{\partial x}\right) = 0$. This implies,

$$\begin{aligned} \frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= \mathsf{a}\ \frac{\partial^2 g_1(x,t)}{\partial x^2}, \\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= \mathsf{b}\ \frac{\partial^2 g_2(x,t)}{\partial x^2} + \mathsf{c}\ g_1(x,t) - \mathsf{d}\ g_2(x,t), \quad x \in \mathbb{R},\ t > 0,\ 0 < \lambda \leq 1 \end{aligned} \tag{4.7}$$

subject to the conditions,

$$\begin{aligned} g_1(x,0) &= m_1 e^{-x^2}, \\ g_2(x,0) &= m_2 e^{-x^2}. \end{aligned} \tag{4.8}$$

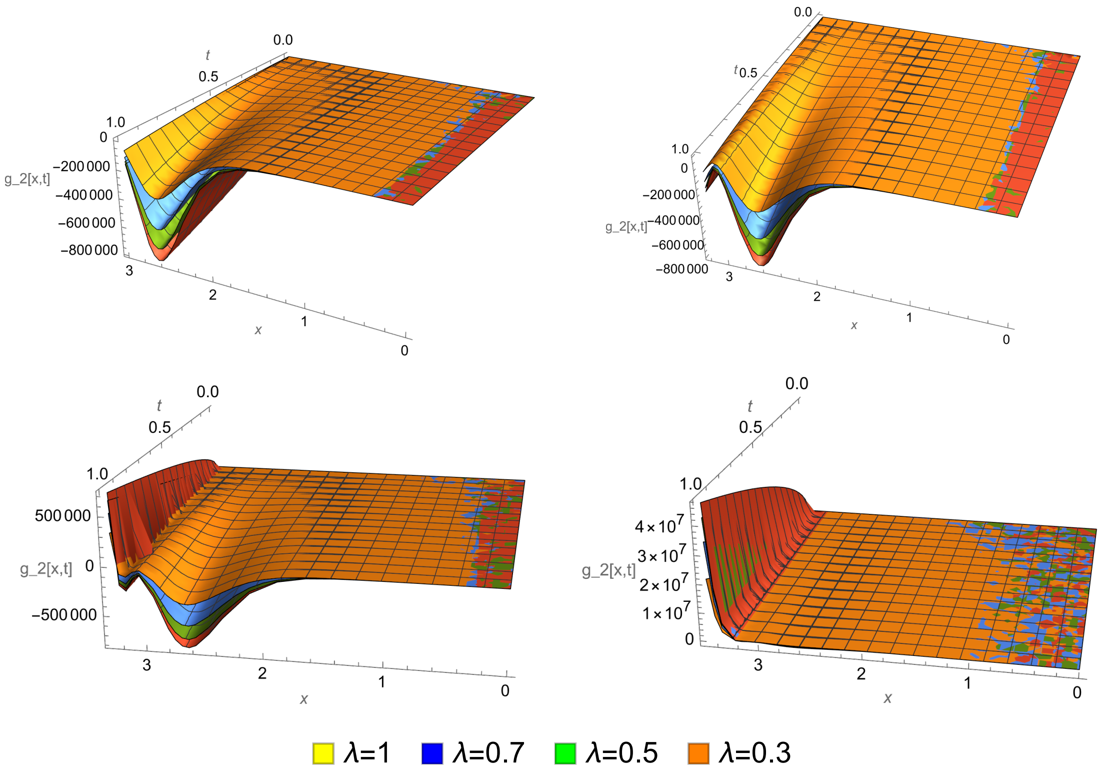

Figure 4.2: Approximate solution $g_2(x,t)$ of the system (4.1)-(4.2) in different spatial frames.

Simplifying the system (4.7)-(4.8) using Formable integral transform, we get,

$$\mathsf{F}\left[g_1(x,t)\right] = m_1 e^{-x^2} + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{a}\ \frac{\partial^2 g_1(x,t)}{\partial x^2}\right]\right],$$

$$\mathsf{F}\left[g_2(x,t)\right] = m_2 e^{-x^2} + \frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{b}\ \frac{\partial^2 g_2(x,t)}{\partial x^2} + \mathsf{c}\ g_1(x,t) - \mathsf{d}\ g_2(x,t)\right]\right].$$

Consequently, by applying the inverse Formable Integral Transform, the coupled equations (4.7)-(4.8) are transformed into the following set of functional equations,

$$\begin{aligned} g_1(x,t) &= m_1 e^{-x^2} + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{a}\ \frac{\partial^2 g_1(x,t)}{\partial x^2}\right]\right]\right] \\ &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[g_1, g_2] \end{aligned}$$

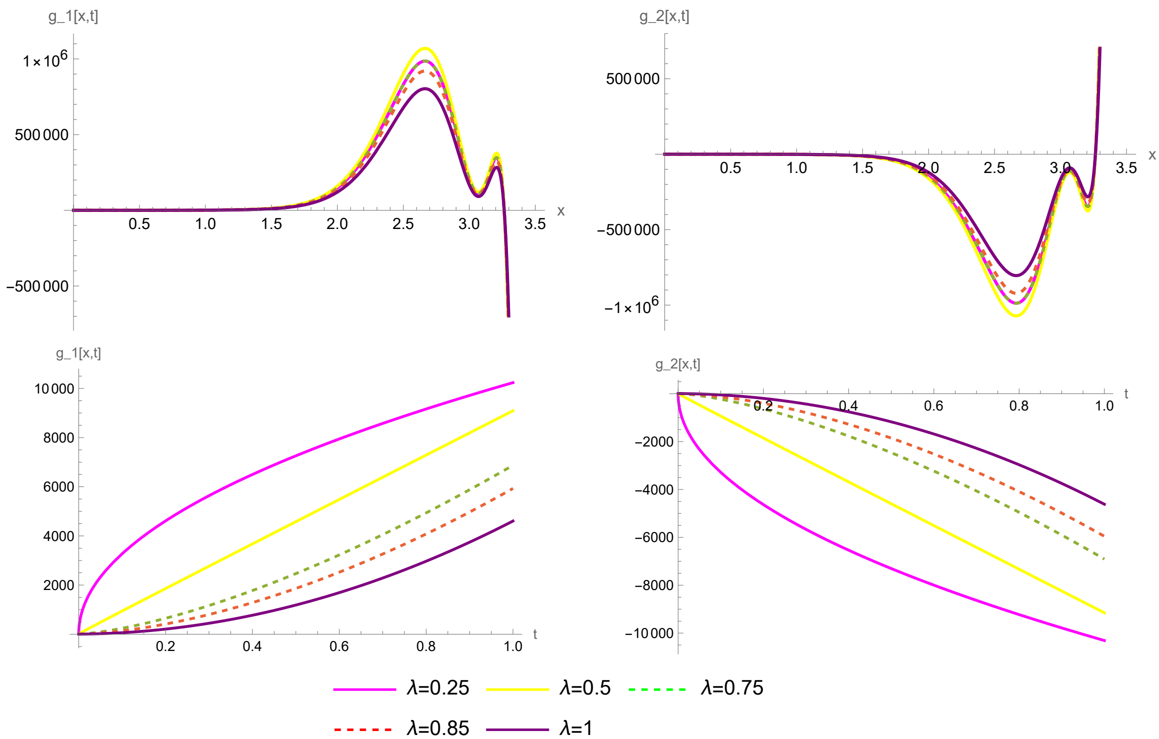

Figure 4.3: Nature of concentrations of the system (4.1)-(4.2) at fixed time ($t = 1.5$) and space ($x = 1.5$).

$$g_2(x,t) = m_2 e^{-x^2} + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{b}\ \frac{\partial^2 g_2(x,t)}{\partial x^2} + \mathsf{c}\ g_1(x,t) - \mathsf{d}\ g_2(x,t)\right]\right]\right]$$
$$= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[g_1, g_2].$$

From the perspective of FTIA, we can express the solution, $g_i(x,t) = \lim_{k\to\infty} \nu_{ik}$, where $\nu_{ik} = \sum_{r=0}^{k} g_{ir}(x,t)$, $1 \le i \le 2$, for which the terms are calculated via iterative scheme (3.5)-(3.6) as follows,

$$\nu_{10} = m_1 e^{-x^2},$$
$$\nu_{20} = m_2 e^{-x^2},$$
$$\nu_{11} = m_1 e^{-x^2} + \frac{2\mathsf{a} m_1 e^{-x^2}\left(2x^2-1\right)t^\lambda}{\Gamma(\lambda+1)},$$
$$\nu_{21} = m_2 e^{-x^2} + \frac{e^{-x^2} t^\lambda \left(\mathsf{c} m_1 - m_2\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right)\right)}{\Gamma(\lambda+1)},$$

$$\nu_{12} = m_1 e^{-x^2}\left(1 + \frac{2\mathsf{a}\left(2x^2-1\right)t^{\lambda}}{\Gamma(\lambda+1)} + \frac{4\mathsf{a}^2\left(4x^4-12x^2+3\right)t^{2\lambda}}{\Gamma(2\lambda+1)}\right),$$

$$\nu_{22} = e^{-x^2}\left(\begin{array}{c} m_2 + \frac{t^{\lambda}\left(\mathsf{c}m_1 - m_2\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right)\right)}{\Gamma(\lambda+1)} + \\ \frac{t^{2\lambda}\left(2\mathsf{ac}m_1\left(2x^2-1\right)+4\mathsf{b}^2 m_2\left(4\left(x^2-3\right)x^2+3\right)+2\mathsf{b}\left(2x^2-1\right)(\mathsf{c}m_1-2\mathsf{d}m_2)+\mathsf{d}(\mathsf{d}m_2-\mathsf{c}m_1)\right)}{\Gamma(2\lambda+1)} \end{array}\right),$$

$$\vdots$$

Thus, the solution of the given system is

$$g_1(x,t) = m_1 e^{-x^2}\left(\begin{array}{c} 1 + \frac{2\mathsf{a}\left(2x^2-1\right)t^{\lambda}}{\Gamma(\lambda+1)} + \frac{4\mathsf{a}^2\left(4x^4-12x^2+3\right)t^{2\lambda}}{\Gamma(2\lambda+1)} + \frac{8\mathsf{a}^3\left(8x^6-60x^4+90x^2-15\right)t^{3\lambda}}{\Gamma(3\lambda+1)} \\ + \frac{16\mathsf{a}^4\left(16x^8-224x^6+840x^4-840x^2+105\right)t^{4\lambda}}{\Gamma(4\lambda+1)} + \cdots \end{array}\right),$$

$$g_2(x,t) = e^{-x^2}\times$$

$$\left(\begin{array}{c} m_2 + \frac{\mathsf{c}m_1 t^{\lambda}}{\Gamma(\lambda+1)} - \frac{m_2 t^{\lambda}\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right)}{\Gamma(\lambda+1)} \\ + \frac{t^{2\lambda}\left(2\mathsf{ac}m_1\left(2x^2-1\right)+4\mathsf{b}^2 m_2\left(4x^4-12x^2+3\right)+2\mathsf{b}\left(2x^2-1\right)(\mathsf{c}m_1-2\mathsf{d}m_2)+\mathsf{d}(\mathsf{d}m_2-\mathsf{c}m_1)\right)}{\Gamma(2\lambda+1)} \\ + \frac{1}{\Gamma(3\lambda+1)}t^{3\lambda}\left(\begin{array}{c} 4\mathsf{a}^2\mathsf{c}m_1\left(4x^4-12x^2+3\right) \\ +2\mathsf{ac}m_1\left(b\left(8x^4-24x^2+6\right)-2\mathsf{d}x^2+\mathsf{d}\right) \\ +8\mathsf{b}^3 m_2\left(8x^6-60x^4+90x^2-15\right) \\ +4\mathsf{b}^2\left(4x^4-12x^2+3\right)(\mathsf{c}m_1-3\mathsf{d}m_2) \\ +2\mathsf{bd}\left(2x^2-1\right)(3\mathsf{d}m_2-2\mathsf{c}m_1)+\mathsf{d}^2(\mathsf{c}m_1-\mathsf{d}m_2) \end{array}\right) \\ + \frac{t^{4\lambda}}{\Gamma(4\lambda+1)}\left(\begin{array}{c} 8\mathsf{a}^3\mathsf{c}m_1\left(8x^6-60x^4+90x^2-15\right) \\ +16\mathsf{b}^4 m_2\left(16x^8-224x^6+840x^4-840x^2+105\right) \\ +4a^2\mathsf{c}m_1\left(2\mathsf{b}\left(8x^6-60x^4+90x^2-15\right)+\mathsf{d}\left(-4x^4+12x^2-3\right)\right) \\ +8\mathsf{b}^3\left(8x^6-60x^4+90x^2-15\right)(\mathsf{c}m_1-4\mathsf{d}m_2) \\ +12\mathsf{b}^2\mathsf{d}\left(4x^4-12x^2+3\right)(2\mathsf{d}m_2-\mathsf{c}m_1) \\ +2\mathsf{bd}^2\left(2x^2-1\right)(3\mathsf{c}m_1-4\mathsf{d}m_2)+\mathsf{d}^3(\mathsf{d}m_2-\mathsf{c}m_1) \\ +2\mathsf{ac}m_1\left(\begin{array}{c} 4\mathsf{b}^2\left(8x^6-60x^4+90x^2-15\right) \\ -4\mathsf{bd}\left(4x^4-12x^2+3\right)+\mathsf{d}^2\left(2x^2-1\right) \end{array}\right) \end{array}\right) \\ + \cdots \end{array}\right).$$

Further, setting $\lambda = 1$ yields the exact solution of the system,

$$g_1(x,t) = m_1 e^{-x^2}\left(\begin{array}{c} 1 + 2\mathsf{a}^2 t^2\left(4x^4-12x^2+3\right) + \frac{4}{3}\mathsf{a}^3 t^3\left(8x^6-60x^4+90x^2-15\right) \\ +\frac{2}{3}\mathsf{a}^4 t^4\left(16x^8-224x^6+840x^4-840x^2+105\right) \\ +2\mathsf{a}t\left(2x^2-1\right)+\cdots \end{array}\right),$$

$$g_2(x,t) = e^{-x^2}\begin{pmatrix} m_2 + \mathsf{c}m_1 t - m_2 t\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right) \\ +\frac{1}{2}t^2\begin{pmatrix} 2\mathsf{ac}m_1\left(2x^2-1\right)+4\mathsf{b}^2 m_2\left(4x^4-12x^2+3\right) \\ +2\mathsf{b}\left(2x^2-1\right)\left(\mathsf{c}m_1-2\mathsf{d}m_2\right)+\mathsf{d}(\mathsf{d}m_2-\mathsf{c}m_1)\end{pmatrix} \\ +\frac{1}{6}t^3\begin{pmatrix} 4\mathsf{a}^2\mathsf{c}m_1\left(4x^4-12x^2+3\right) \\ +2\mathsf{ac}m_1\left(\mathsf{b}\left(8x^4-24x^2+6\right)-2\mathsf{d}x^2+\mathsf{d}\right) \\ +8\mathsf{b}^3 m_2\left(8x^6-60x^4+90x^2-15\right) \\ +4\mathsf{b}^2\left(4x^4-12x^2+3\right)\left(\mathsf{c}m_1-3\mathsf{d}m_2\right) \\ +2\mathsf{bd}\left(2x^2-1\right)\left(3\mathsf{d}m_2-2\mathsf{c}m_1\right)+\mathsf{d}^2(\mathsf{c}m_1-\mathsf{d}m_2)\end{pmatrix}+\cdots \end{pmatrix}.$$

We observe that the derived series exhibits a high degree of consistency with HP-STM series detailed in [27] and with FCT series in [18]. The visual data presented in the figures will further elucidate the characteristics of cell intensity and chemical signals.

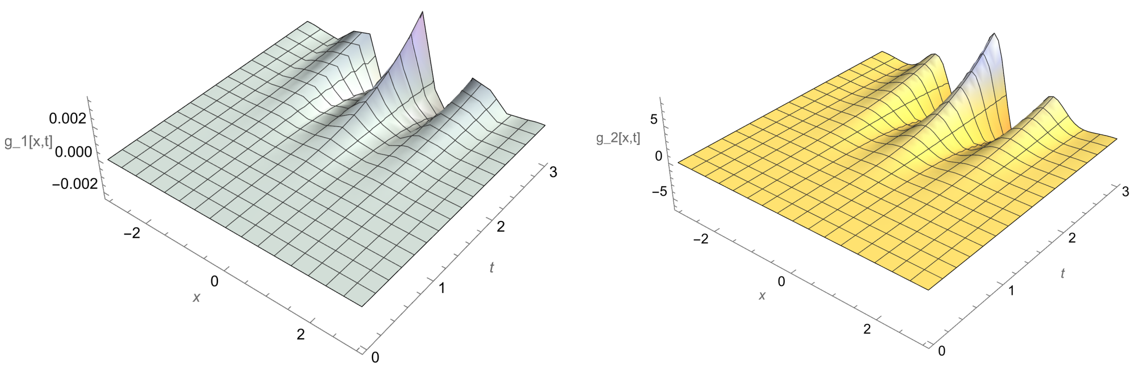

Figure 4.4: The five term approximate series solutions of the system (4.7)-(4.8) for $\lambda = 1$.

Example 4.3. Let us take sensitivity function $\psi(g_2) = g_2$ in (1.3)-(1.4), then we have,
$\dfrac{\partial}{\partial x}\left(g_1.\dfrac{\partial g_2}{\partial x}\right) = \dfrac{\partial g_1}{\partial x}.\dfrac{\partial g_2}{\partial x} + g_1\dfrac{\partial^2 g_2}{\partial x^2}$. This implies,

$$\begin{aligned} \frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= \mathsf{a}\ \frac{\partial^2 g_1}{\partial x^2} - \frac{\partial g_1}{\partial x}.\frac{\partial g_2}{\partial x} - g_1\frac{\partial^2 g_2}{\partial x^2}, \\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= \mathsf{b}\ \frac{\partial^2 g_2}{\partial x^2} + \mathsf{c}\ g_1 - \mathsf{d}\ g_2, \quad x\in\mathbb{R},\ t>0,\ 0<\lambda\leq 1 \end{aligned} \tag{4.9}$$

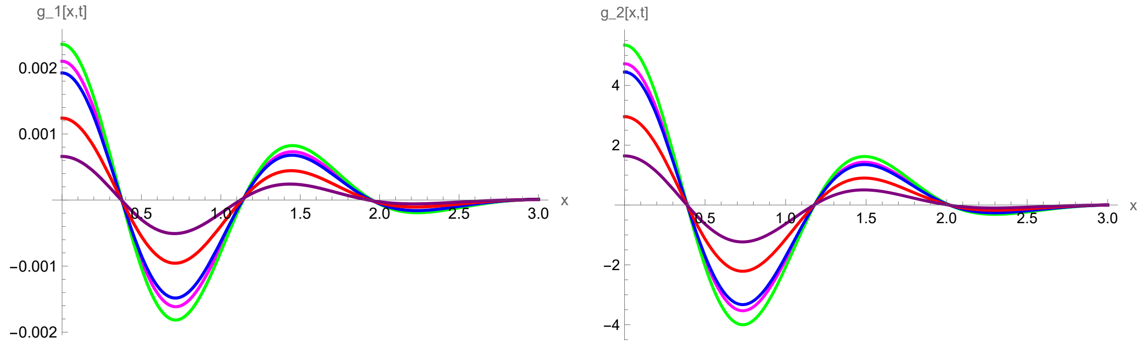

Figure 4.5: 2D view of chemoattractent process of the system (4.1)-(4.2) captured at $t = 2$.

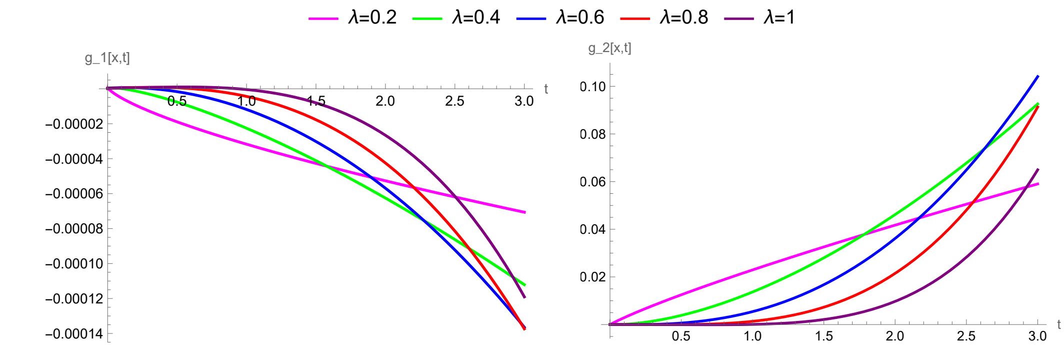

Figure 4.6: 2D view of chemoattractent process of the system (4.7)-(4.8) captured at $x = 2$.

subject to the conditions,

$$\begin{aligned} g_1(x,0) &= m_1 e^{-x^2}, \\ g_2(x,0) &= m_2 e^{-x^2}. \end{aligned} \tag{4.10}$$

Using FTIA, the IVP (4.9)-(4.10) can be transformed into

$$\begin{aligned} g_1(x,t) &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[g_1,g_2] + \mathcal{N}_1^*[g_1,g_2], \\ g_2(x,t) &= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[g_1,g_2], \end{aligned}$$

where,

$$\begin{aligned} \mathcal{H}_1^* &= m_1 e^{-x^2}, \\ \mathcal{H}_2^* &= m_2 e^{-x^2}, \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_1^* &= \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{a}\ \frac{\partial^2 g_1}{\partial x^2}\right]\right]\right]\\
\mathcal{L}_2^* &= \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{b}\ \frac{\partial^2 g_2(x,t)}{\partial x^2}+\mathsf{c}\ g_1(x,t)-\mathsf{d}\ g_2(x,t)\right]\right]\right]\\
\mathcal{N}_1^* &= \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[-\frac{\partial g_1}{\partial x}.\frac{\partial g_2}{\partial x}-g_1\frac{\partial^2 g_2}{\partial x^2}\right]\right]\right]
\end{aligned}
$$

By employing iterative procedure (3.5)-(3.6), we have,

$$
\begin{aligned}
\nu_{10} &= m_1 e^{-x^2},\\
\nu_{20} &= m_2 e^{-x^2},\\
\nu_{11} &= m_1 e^{-x^2}+\frac{2m_1 e^{-2x^2}t^\lambda\left(\mathsf{a}e^{x^2}\left(2x^2-1\right)-4m_2x^2+m_2\right)}{\Gamma(\lambda+1)},\\
\nu_{21} &= m_2 e^{-x^2}+\frac{e^{-x^2}t^\lambda\left(\mathsf{c}m_1-m_2\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right)\right)}{\Gamma(\lambda+1)},
\end{aligned}
$$

Hence, the three term approximation for the system (4.9)-(4.10) is given by,

$$
\begin{aligned}
\nu_{12} = m_1 e^{-x^2} &+ \frac{2m_1 e^{-2x^2}t^\lambda\left(\mathsf{a}e^{x^2}\left(2x^2-1\right)-4m_2x^2+m_2\right)}{\Gamma(\lambda+1)}\\
&- \frac{2m_1 e^{-2x^2}t^{2\lambda}\begin{pmatrix} m_2\left(-4x^2(29\mathsf{a}+9\mathsf{b}+\mathsf{d})+16x^4(5\mathsf{a}+b)+14\mathsf{a}+6\mathsf{b}+\mathsf{d}\right)\\ +\mathsf{c}m_1\left(4x^2-1\right)-2m_2^2e^{-x^2}\left(24x^4-18x^2+1\right)\\ -2\mathsf{a}^2e^{x^2}\left(4\left(x^2-3\right)x^2+3\right)\end{pmatrix}}{\Gamma(2\lambda+1)}+\cdots,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\nu_{22} = m_2 e^{-x^2} &+ \frac{e^{-x^2}t^\lambda\left(\mathsf{c}m_1-m_2\left(\mathsf{b}\left(2-4x^2\right)+\mathsf{d}\right)\right)}{\Gamma(\lambda+1)}\\
&+ \frac{e^{-2x^2}t^{2\lambda}\begin{pmatrix} 2\mathsf{ac}m_1e^{x^2}\left(2x^2-1\right)+4\mathsf{b}^2m_2e^{x^2}\left(4x^4-12x^2+3\right)\\ +2\mathsf{b}e^{x^2}\left(2x^2-1\right)\left(\mathsf{c}m_1-2\mathsf{d}m_2\right)-\mathsf{cd}m_1e^{x^2}\\ -8\mathsf{c}m_1m_2x^2+2\mathsf{c}m_1m_2+\mathsf{d}^2m_2e^{x^2}\end{pmatrix}}{\Gamma(2\lambda+1)}+\cdots,
\end{aligned}
\tag{4.12}
$$

We observe that the approximations in equations (4.11) and (4.12) shows good agreement with the series derived by ATIM in [1] and by NDHPM in [17].

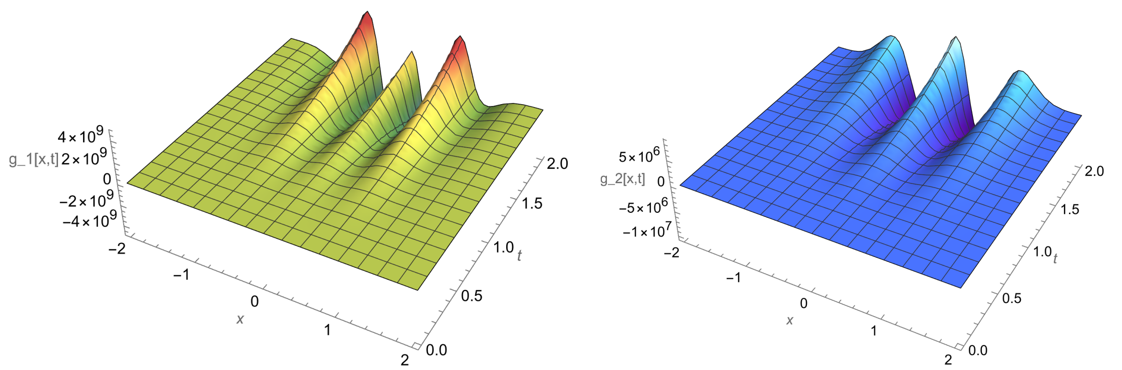

Figure 4.7: 3D view of three term approximate solution of the system (4.9)-(4.10) for $\lambda = 1$.

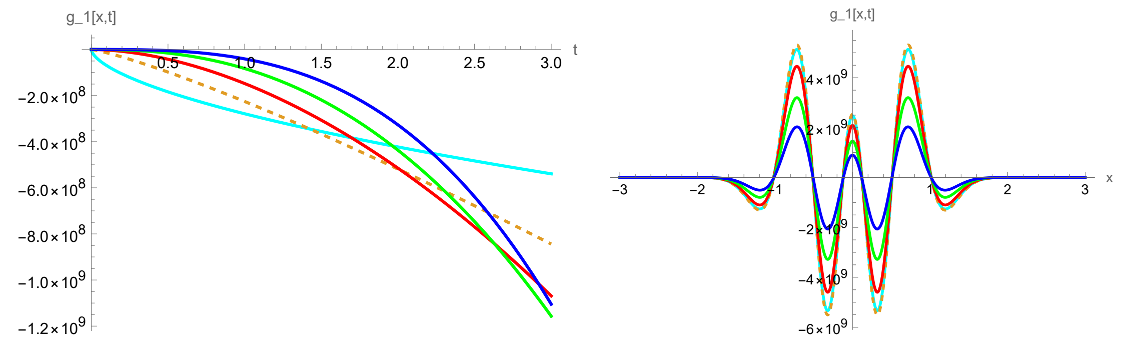

Figure 4.8: 2D view of the solution g_1 of the system (4.9)-(4.10) at fixed time and spatial frames

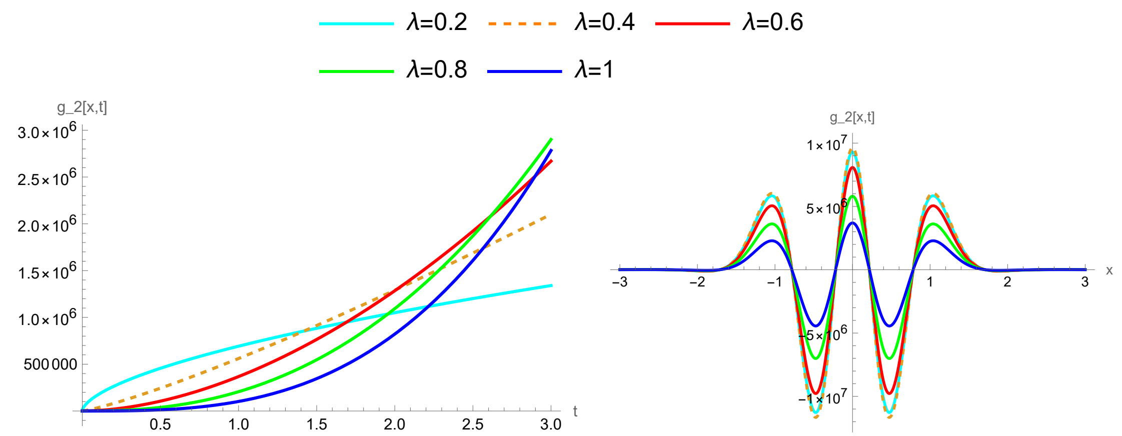

Figure 4.9: 2D view of the solution g_2 of the system (4.9)-(4.10) at fixed time and spatial frames

Example 4.4. If we set $\psi(g_2) = g_2^2$ and $g_{10}(x) = m_1 \sin(x)$, $g_{20}(x) = m_2 \sin(x)$, then the system (1.3)-(1.4) will become,

$$\begin{aligned}\frac{\partial^\lambda g_1(x,t)}{\partial t^\lambda} &= \mathsf{a}\ \frac{\partial^2 g_1}{\partial x^2} - \frac{\partial g_1}{\partial x}.\frac{\partial g_2^2}{\partial x} - g_1 \frac{\partial^2 g_2^2}{\partial x^2},\\ \frac{\partial^\lambda g_2(x,t)}{\partial t^\lambda} &= \mathsf{b}\ \frac{\partial^2 g_2}{\partial x^2} + \mathsf{c}\ g_1 - \mathsf{d}\ g_2, \quad x \in \mathbb{R},\ t > 0,\ 0 < \lambda \leq 1\end{aligned} \tag{4.13}$$

subject to the conditions,

$$\begin{aligned} g_1(x,0) &= m_1 \sin(x),\\ g_2(x,0) &= m_2 \sin(x).\end{aligned} \tag{4.14}$$

The application of the FTIA to the system (4.13)-(4.14) yields the following functional equation:

$$\begin{aligned} g_1(x,t) &= m_1 \sin(x) + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{a}\ \frac{\partial^2 g_1}{\partial x^2}\right]\right]\right] + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[-\frac{\partial g_1}{\partial x}.\frac{\partial g_2^2}{\partial x} - g_1\frac{\partial^2 g_2^2}{\partial x^2}\right]\right]\right]\\ &= \mathcal{H}_1^*(x,t) + \mathcal{L}_1^*[g_1, g_2] + \mathcal{N}_1^*[g_1, g_2]\\ g_2(x,t) &= m_2 \sin(x) + \mathsf{F}^{-1}\left[\frac{v^\lambda}{s^\lambda}\left[\mathsf{F}\left[\mathsf{b}\ \frac{\partial^2 g_2}{\partial x^2} + \mathsf{c}\ g_1 - \mathsf{d}\ g_2\right]\right]\right]\\ &= \mathcal{H}_2^*(x,t) + \mathcal{L}_2^*[g_1, g_2].\end{aligned}$$

Then the approximated series solution is given by $\nu_{ik} = \sum_{r=0}^{k} g_{ir}(x,t),\ 1 \leq i \leq 2$, where,

$$\begin{aligned}\nu_{10} &= m_1 \sin(x),\\ \nu_{20} &= m_2 \sin(x),\\ \nu_{11} &= m_1 \sin(x) - \frac{m_1 t^\lambda \sin(x)\left(\mathsf{a} + 3m_2^2\cos(2x) + m_2^2\right)}{\Gamma(\lambda+1)},\\ \nu_{21} &= m_2 \sin(x) + \frac{t^\lambda \sin(x)(\mathsf{c}m_1 - m_2(\mathsf{b}+\mathsf{d}))}{\Gamma(\lambda+1)},\\ \nu_{12} &= m_1 \sin(x) - \frac{m_1 t^\lambda \sin(x)\left(\mathsf{a} + 3m_2^2\cos(2x) + m_2^2\right)}{\Gamma(\lambda+1)}\\ &+ \frac{m_1 t^{2\lambda}\begin{pmatrix}\sin(x)\left(4\mathsf{a}^2 - 4m_2^2(\mathsf{a}+\mathsf{b}+\mathsf{d}) + 4\mathsf{c}m_1m_2 - 2m_2^4\right)\\ -3m_2\sin(3x)\left(-4m_2(5\mathsf{a}+\mathsf{b}+\mathsf{d}) + 4\mathsf{c}m_1 + m_2^3\right) + 15m_2^4\sin(5x)\end{pmatrix}}{4\Gamma(2\lambda+1)} + \cdots,\end{aligned}$$

$$\nu_{22} = m_2 \sin(x) + \frac{t^{\lambda} \sin(x)(\mathsf{c}m_1 - m_2(\mathsf{b}+\mathsf{d}))}{\Gamma(\lambda+1)} - \frac{t^{2\lambda} \sin(x)\left(\mathsf{c}m_1(\mathsf{a}+\mathsf{b}+\mathsf{d}) - m_2(\mathsf{b}+\mathsf{d})^2 + 3\mathsf{c}m_1 m_2^2 \cos(2x) + \mathsf{c}m_1 m_2^2\right)}{\Gamma(2\lambda+1)} + \cdots,$$

on proceeding with the iteration, we get,

$$g_1(x,t) = m_1 \sin(x) - \frac{m_1 t^{\lambda} \sin(x)\left(\mathsf{a} + 3m_2^2 \cos(2x) + m_2^2\right)}{\Gamma(\lambda+1)} + \frac{m_1 t^{2\lambda} \begin{pmatrix} \sin(x)\left(4\mathsf{a}^2 - 4m_2^2(\mathsf{a}+\mathsf{b}+\mathsf{d}) + 4\mathsf{c}m_1 m_2 - 2m_2^4\right) \\ -3m_2 \sin(3x)\left(-4m_2(5\mathsf{a}+\mathsf{b}+\mathsf{d}) + 4\mathsf{c}m_1 + m_2^3\right) + 15m_2^4 \sin(5x) \end{pmatrix}}{4\Gamma(2\lambda+1)}$$
$$\frac{\Gamma(2\lambda+1)\sin(x)}{4\Gamma(3\lambda+1)} m_1 t^{3\lambda} \times \left(- \frac{4(\mathsf{c}m_1 - m_2(\mathsf{b}+\mathsf{d})) \begin{pmatrix} 3\cos(2x)\left(\mathsf{c}m_1 - m_2\left(2\mathsf{a}+\mathsf{b}+\mathsf{d}+4m_2^2\right)\right) \\ -m_2\left(2\mathsf{a}+\mathsf{b}+\mathsf{d}+5m_2^2\right) \\ +\mathsf{c}m_1 - 15m_2^3 \cos(4x) \end{pmatrix}}{\Gamma(\lambda+1)^2} + \cdots \right) + \cdots, \tag{4.15}$$

$$g_2(x,t) = m_2 \sin(x) + \frac{t^{\lambda} \sin(x)(\mathsf{c}m_1 - m_2(\mathsf{b}+\mathsf{d}))}{\Gamma(\lambda+1)} - \frac{t^{2\lambda} \sin(x)\left(\mathsf{c}m_1(\mathsf{a}+\mathsf{b}+\mathsf{d}) - m_2(\mathsf{b}+\mathsf{d})^2 + 3\mathsf{c}m_1 m_2^2 \cos(2x) + \mathsf{c}m_1 m_2^2\right)}{\Gamma(2\lambda+1)}$$
$$+ \frac{1}{4\Gamma(3\lambda+1)} t^{3\lambda} \times \begin{pmatrix} 2\sin(x) \begin{pmatrix} 2\mathsf{c}m_1\left(\mathsf{a}^2 + \mathsf{a}(\mathsf{b}+\mathsf{d}) + (\mathsf{b}+\mathsf{d})^2\right) \\ -\mathsf{c}m_1 m_2^2(2\mathsf{a} + 3(\mathsf{b}+\mathsf{d})) \\ -2m_2\left((\mathsf{b}+\mathsf{d})^3 - c^2 m_1^2\right) - \mathsf{c}m_1 m_2^4 \end{pmatrix} \\ +3\mathsf{c}m_1 m_2 \begin{pmatrix} \sin(3x)\left(20\mathsf{a}m_2 + 22\mathsf{b}m_2 - 4\mathsf{c}m_1 + 6\mathsf{d}m_2 - m_2^3\right) \\ +5m_2^3 \sin(5x) \end{pmatrix} \end{pmatrix} + \cdots, \tag{4.16}$$

A consistent agreement is observed between our results and the numerical solutions of the chemotaxis model obtained via the LHPM and MHATM methods, as reported in [20].

In the subsequent section, a detailed interpretation of our findings is presented through graphical representation.

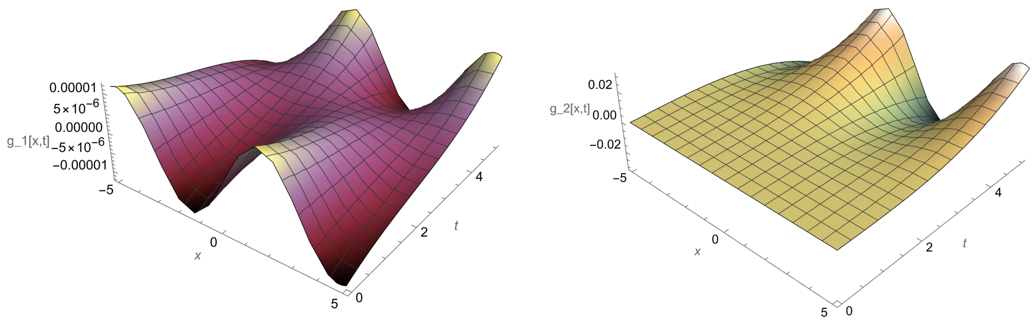

Figure 4.10: Approximate solution of the system (4.13)-(4.14) for $\lambda = 1$.

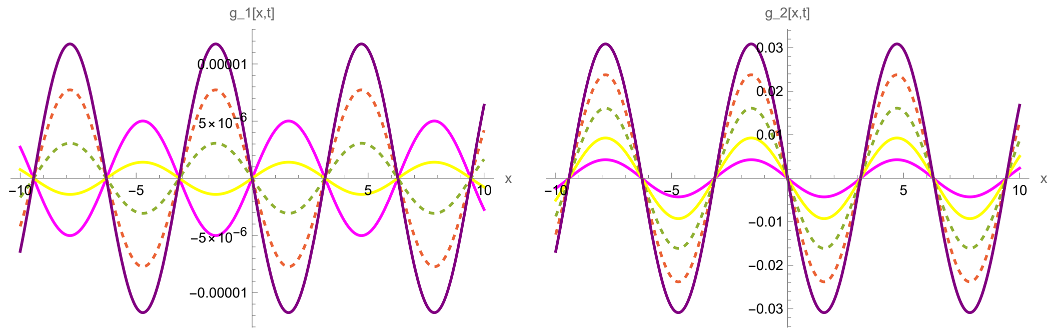

Figure 4.11: Nature of g_1 and g_2 of the system (4.13)-(4.14) at $t = 5$.

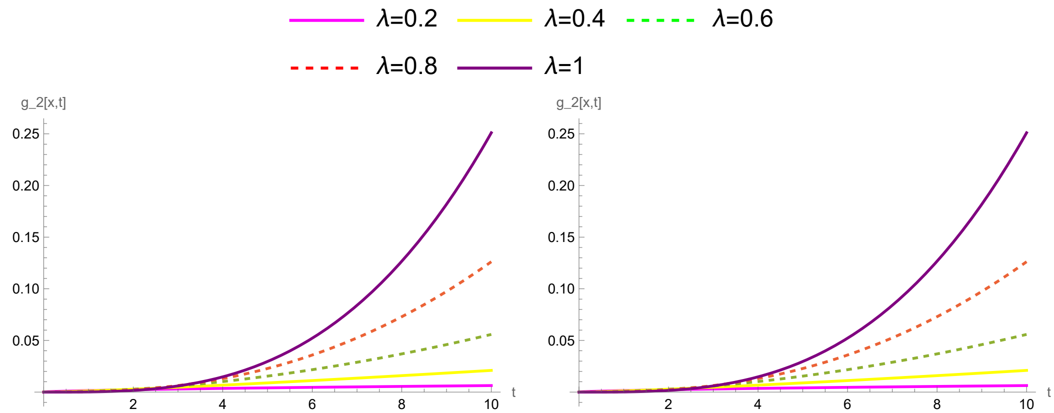

Figure 4.12: Nature of g_1 and g_2 of the system (4.13)-(4.14) at $x = 5$.

5. Discussions and Results

This section meticulously outlines our proposed methodology, providing a solid foundation for the analysis of its implementation and interpretation of the resulting data. For the graphical representation of solutions in Examples 4.2 and 4.4, we choose the following parameters: $\mathsf{a} = 0.5, \mathsf{b} = 3, \mathsf{c} = 1, \mathsf{d} = 2, m_1 = 0.000012, m_2 = 0.000016$. For example 4.3, the parameters are $\mathsf{a} = 0.5, \mathsf{b} = 3, \mathsf{c} = 1, \mathsf{d} = 0.8, m_1 = 160, m_2 = 120$.

The three-dimensional and two-dimensional views presented in figures (4.1)-(4.2) and (4.3), clearly illustrate the dynamic behaviour of the concentrations of the chemical Schnakenberg model (4.1)-(4.2). Specifically, Fig.4.1 demonstrates an initial increase followed by a decrease in g_1 concentration, while Fig.4.2 shows a corresponding initial drop and subsequent increase in g_2 concentration.

The surface plots illustrating the approximate solution of the FKS model under different sensitivity functions are presented in figures 4.4, 4.7 and 4.10 respectively. These 3-D visualizations reveal how cell intensity and the chemical signal levels vary across the spatiotemporal frame.

The 2-D plot of chemoattractant process with respect to x and t, governed by the system (4.7)-(4.8) are given in figure 4.5 and 4.6. These plots are mapped for different order of derivatives which potentially reveal the cell aggregation, chemoattractant gradient and other features. Figure 4.8 and 4.9 respectively provide a two-dimensional insight into the effect of order of derivative (λ) on the pattern of g_1 and g_2 of the system (4.9)-(4.10), evaluated at $t = 1.5$ and $x = 1.5$. The temporal behaviour (at $x = 5$) and the spatial behaviour (at $t = 5$) of the solutions (4.15) and (4.16) of the FKS model with sensitivity $\psi(g_2) = g_2^2$ are shown in figures (4.11) and (4.12).

These graphical representations clearly illustrate that as the fractional order approaches integer value, the distinct curves representing these solutions visibly merge with the curve of the exact solution, lending further validity to our proposed method. Moreover, the significant correlation demonstrated between the exact solution and the approximations derived using FTIA underscores its high accuracy and efficiency, Furthermore, our results are in close harmony with those obtained by several existing methods in the literature, including LHAM, NDT, HPSTM, ATIM and MHATM. Also, we note that this harmony is achieved with minimal iterations and, notably, without the need for parameter guessing or optimization, highlighting a key advantage of FTIA approach.

6. Conclusion

A novel and straightforward approach, termed the Formable Transform Iterative Algorithm (FTIA), has been introduced to obtain approximate series solutions for nonlinear fractional differential equations. The efficacy of this method has been validated through its application to two significant fractional reaction-transport systems. Graphical analyses reveal consistency between our results and those obtained by established methods in the literature, thereby enhancing the comprehension of chemotactic behavior in biological contexts. Furthermore, FTIA demonstrates rapid convergence with minimal computational effort and does not require parameter tuning, establishing it as a robust and efficient alternative for solving a broad class of nonlinear fractional models.

Acknowledgment

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